

Spectral expansion of Schwartz linear operators

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Abstract

In this paper we prove and apply a theorem of spectral expansion for Schwartz linear operators which have a Schwartz linearly independent eigenfamily. This type of spectral expansion is the analogous of the spectral expansion for self-adjoint operators of separable Hilbert spaces, but in the case of eigenfamilies of vectors indexed by the real Euclidean spaces. The theorem appears formally identical to the spectral expansion in the finite dimensional case, but for the presence of continuous superpositions instead of finite sums. The Schwartz expansion we present is one possible rigorous and simply manageable mathematical model for the spectral expansions used frequently in Quantum Mechanics, since it appears in a form extremely similar to the current formulations in Physics.

1 Preliminaries

In the following we shall use (with Dieudonné) the notation $\mathcal{L}(\mathcal{S}'_n)$ for the space of \mathcal{S} -linear endomorphisms on the space \mathcal{S}'_n , instead of $\mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_n)$, it is just the space of continuous linear endomorphisms with respect to the weak* topology (or, equivalently, with respect to the strong* topology) on the distribution space \mathcal{S}'_n .

Let E be a vector space and let A be a linear operator of E into E . The set of all the eigenvectors of the operator A is denoted (following Dieudonné)

by $E(A)$. The set of all the eigenvalues of the operator A is denoted by $\text{ev}(A)$. Moreover, the eigenspace corresponding to an eigenvalue $a \in \mathbb{K}$ is denoted (following Dieudonné) by $E_a(A)$. For every eigenvector u of the operator A , there is only one eigenvalue a such that $A(u) = au$, so that we can consider the projection

$$e_A : E(A) \rightarrow \text{ev}(A)$$

associating with every eigenvector u of the operator A its eigenvalue, so that $e_A(u)$ is the unique scalar such that

$$A(u) = e_A(u)u.$$

It is clear that the set $E_a^\neq(A)$, collection of all eigenvectors of A corresponding to the eigenvalue a , coincides with the reciprocal image $e_A^{-1}(a)$. So that we have constructed a fiber space $(E(A), \text{ev}(A), e_A)$, with support $E(A)$, basis $\text{ev}(A)$ and projection e_A .

Remark. We can obtain a fiber vector space, considering the set E_A of all pairs (a, u) in the Cartesian product $\mathbb{K} \times E$ with a eigenvalue of the operator A and such that $A(u) = au$ (so that the partial zero-element $(a, 0)$ lies in E_A , for every eigenvalue a of A) with projection

$$e_A : E_A \rightarrow \text{ev}(A) : (a, u) \mapsto a,$$

and basis $\text{ev}(A)$. Each fiber $e_A^{-1}(a)$ of the fiber space $(E_A, \text{ev}(A), e_A)$ is isomorphic (in the obvious way) with the eigenspace $E_a(A)$ (by the isomorphism $I_a : e_A^{-1}(a) \rightarrow E_a(A)$ sending the pair (a, u) into the vector u).

2 Spectral \mathcal{S} expansions

Definition (of eigenfamily). Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an \mathcal{S} linear endomorphism on the space \mathcal{S}'_n , i.e. a continuous linear endomorphism on the space \mathcal{S}'_n , let $a \in \mathcal{O}_M^{(m)}$ be a complex smooth slowly increasing function defined on the Euclidean space \mathbb{R}^m and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a Schwartz family in \mathcal{S}'_n . We say that the Schwartz family v is an \mathcal{S} **eigenfamily of the operator A with respect to the system of eigenvalues a** if, for each index $p \in \mathbb{R}^m$, the vector v_p is an eigenvector of the operator A with respect to the eigenvalue

$a(p)$. In other terms, the family v is an \mathcal{S} eigenfamily of the operator A , with respect to the system of eigenvalues a , if, for each point index $p \in \mathbb{R}^m$, we have

$$A(v_p) = a(p)v_p,$$

so that, in terms of families, the image family $A(v)$ can be written as the product $A(v) = av$, of the family v times the function a (recall that the space of Schwartz families $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is a module over the ring of slowly increasing smooth functions $\mathcal{O}_M^{(m)}$).

Now we can state and prove the principal theorem on spectral Schwartz expansion.

Theorem (of \mathcal{S} spectral expansion). *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an \mathcal{S} linear endomorphism, let $a \in \mathcal{O}_M^{(m)}$ be a smooth slowly increasing function defined on the Euclidean space \mathbb{R}^m and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an \mathcal{S} linearly independent Schwartz eigenfamily of the operator A , with respect to the system of eigenvalues a . Then, we have the spectral \mathcal{S} expansion*

$$A(u) = \int_{\mathbb{R}^m} (a[u|v])v.$$

for each tempered distribution u in the \mathcal{S} linear hull $\mathcal{S}\text{span}(v)$ of the eigenfamily v , where $[u|v]$ is the coordinate distribution of the vector u with respect to the Schwartz basis v .

Proof. For each distribution u in the \mathcal{S} linear hull $\mathcal{S}\text{span}(v)$ of the eigenfamily v , we have

$$\begin{aligned} A(u) &= A\left(\int_{\mathbb{R}^m} [u|v]v\right) = \\ &= \int_{\mathbb{R}^m} [u|v]A(v) = \\ &= \int_{\mathbb{R}^m} [u|v](av) = \\ &= \int_{\mathbb{R}^m} (a[u|v])v. \end{aligned}$$

In fact, the third equality holds because of definition of pointwise product of a smooth slowly increasing function by a Schwartz family, i.e.

$$\begin{aligned} A(v)_p &= A(v_p) = \\ &= a(p)v_p = \\ &= (av)(p), \end{aligned}$$

for every point index p of the family v , as we already have noted; and the fourth equality holds because of elementary properties of the product of a smooth slowly increasing function by a linear continuous operator; indeed, for every test function ϕ , we have

$$\begin{aligned} \left(\int_{\mathbb{R}^m} [u|v](av) \right) (\phi) &= [u|v]((av)^\wedge(\phi)) = \\ &= [u|v](a\widehat{v}(\phi)) = \\ &= (a[u|v])(\widehat{v}(\phi)) = \\ &= \left(\int_{\mathbb{R}^m} (a[u|v])v \right) (\phi), \end{aligned}$$

as we well know in the general case; this concludes the proof. ■

Remind (superposition of a Schwartz family with respect to an operator). Recall the definition of superposition of an \mathcal{S} family with respect to an operator. Let V be a subspace of the distribution space \mathcal{S}'_n , let $A \in \text{Hom}(V, \mathcal{S}'_m)$ be a linear operator and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an \mathcal{S} family of distributions in \mathcal{S}'_n . The superposition of the family v with respect to the operator A , is the operator defined as it follows

$$\int_{\mathbb{R}^m} Av : V \rightarrow \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

Remark. So, in the conditions of the above theorem, if V is the Schwartz linear hull of the family v , we can write

$$A|_V = \int_{\mathbb{R}^m} a[.|v] v,$$

saying that the (domain) restriction $A|_V$ of the operator A to the \mathcal{S} linear hull of the family v is the superposition of the family v with respect to the product of the coordinate operator

$$[.|v] : V \rightarrow \mathcal{S}'_m : u \mapsto [u|v] = ({}^t\widehat{v}|_{(\mathcal{S}'_m, V)})^{-1}(u),$$

of the family v by the system of eigenvalues a .

Remark (on the resolution of identity). The above theorem generalizes the *Resolution of Identity theorem*. Indeed, every \mathcal{S} basis of the space \mathcal{S}'_n is an \mathcal{S} eigenfamily of the identity operator $(\cdot)_{\mathcal{S}'_n}$ of the space \mathcal{S}'_n , with respect to the constant unitary system of eigenvalues $1_{\mathbb{R}^m}$, so that we have

$$(\cdot)_{\mathcal{S}'_n} = \int_{\mathbb{R}^m} [\cdot|v]v,$$

for every \mathcal{S} basis v of the space \mathcal{S}'_n . Moreover, if j_V is the canonical injection of the Schwartz linear hull V of an \mathcal{S} linearly independent family v into \mathcal{S}'_n , we have

$$j_V = \int_{\mathbb{R}^m} [\cdot|v]v,$$

where \mathbb{R}^m is the index set of the family v , since the canonical injection j_V is just the (domain) restriction to the hull V of the identity operator $(\cdot)_{\mathcal{S}'_n}$ of the space \mathcal{S}'_n .

Remark (on Schwartz diagonalizable operator). The above theorem holds in the particular case in which there is an \mathcal{S} basis of the space \mathcal{S}'_n formed by eigenvectors of the operator A . This case is the basic theme of the theory of Schwartz diagonalizable operators.

3 Expansions by spectral distributions

We recall that:

- if A is a linear continuous endomorphism of a Hilbert space H , a **spectral measure of the operator** A is a linear continuous operator μ from the space $C^0(S)$, of continuous complex functions defined on the spectrum S of the operator A , into the space $\mathcal{L}(H)$, of linear continuous endomorphisms of the Hilbert space H , such that the operator A can be seen in the integral form

$$A = \mu(j_S) = \int_S j_S \mu,$$

where j_S is the canonical injection of the spectrum S of the operator A into the complex field.

We shall obtain some similar spectral expansions for Schwartz-linear operators admitting a Schwartz eigenbasis of the space.

In the conditions of our spectral expansion theorem (for simplicity when v is a Schwartz basis of the entire space), we have

$$A = \int_{\mathbb{R}^m} (a[\cdot|v])v,$$

where a is the smooth slowly increasing system of eigenvalues of the operator A corresponding to the Schwartz eigensystem v .

3.1 Spectral distribution of a Schwartz basis

- **Position of the problem.** Let v be a Schwartz basis of the space \mathcal{S}'_n and consider the linear operator

$$\mu_v : \mathcal{O}_M^{(m)} \rightarrow \mathcal{L}(\mathcal{S}'_n)$$

defined by

$$\mu_v(f) = \int_{\mathbb{R}^m} (f[\cdot|v])v,$$

for every function f in $\mathcal{O}_M^{(m)}$. The operator μ_v will be our *generalized spectral measure* (or, better, our *spectral distribution*) capable to expand (in the sense of spectral measure) each operator A admitting v as a Schwartz eigenbasis.

Definition (the generalized spectral measure of a Schwartz basis). *We call the above operator μ_v the generalized spectral measure of the Schwartz basis v .*

Note, first of all, that this operator μ_v is continuous, with respect to the standard topology of the function space $\mathcal{O}_M^{(m)}$ and the pointwise topology

of the operator space $\mathcal{L}(\mathcal{S}'_n)$, since it is compositions of linear continuous operators. Namely, it is the composition of the operator chain

$$\mathcal{O}_M^{(m)} \rightarrow \mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m) \rightarrow \mathcal{L}(\mathcal{S}'_n),$$

defined by

$$f \mapsto f[.]v \mapsto \int_{\mathbb{R}^m} (f[.]v)v.$$

Example (the spectral distribution of the Dirac basis). Let us consider the Dirac family δ , we have

$$\begin{aligned} \mu_\delta(f)(u) &= \int_{\mathbb{R}^m} (f[u|\delta])\delta = \\ &= fu = \\ &= M_f(u), \end{aligned}$$

for every distribution u , so that the spectral distribution of the Dirac family δ is the operator $M : \mathcal{O}_M^{(n)} \rightarrow \mathcal{L}(\mathcal{S}'_n)$ sending each slowly increasing function f of $\mathcal{O}_M^{(n)}$ into its multiplication operator $M_f \in \mathcal{L}(\mathcal{S}'_n)$.

We note the following trivial but meaningful properties.

Proposition. *The spectral distribution μ_v of a Schwartz linearly independent family v is an algebraic linear immersion of the space $\mathcal{O}_M^{(m)}$ into the space $\mathcal{L}(\mathcal{S}'_n)$.*

Proof. Indeed if $\mu_v(f) = \mu_v(g)$, then

$$\int_{\mathbb{R}^m} (f[.]v)v = \int_{\mathbb{R}^m} (g[.]v)v,$$

by the Schwartz linear independence of v we deduce

$$f[.]v = g[.]v,$$

and so, for every index p of the family v , we have

$$f[v_p|v] = g[v_p|v],$$

that is equivalent to

$$f\delta_p = g\delta_p,$$

and so, $f(p) = g(p)$ for every p in \mathbb{R}^m : we have the equality $f = g$. ■

Proposition. *The operator $\mu_v(f)$ has v as an eigenbasis and f as the system of eigenvalues corresponding to v .*

Proof. Indeed if $\mu_v(f)(v_p) = \mu_v(g)$, then

$$\begin{aligned} \mu_v(f)(v_p) &= \int_{\mathbb{R}^m} (f[v_p|v])v = \\ &= \int_{\mathbb{R}^m} (f\delta_p)v = \\ &= \int_{\mathbb{R}^m} f(p)\delta_p v = \\ &= f(p) \int_{\mathbb{R}^m} \delta_p v = \\ &= f(p)v_p, \end{aligned}$$

for every p in \mathbb{R}^m : we have so

$$\mu_v(f)(v) = fv.$$

as we claimed. ■

The first above property is contained in the following more complete result.

Proposition (of algebra homomorphism). *The spectral distribution μ_v of a Schwartz basis v is an injective homomorphism of the function algebra $\mathcal{O}_M(\mathbb{R}^m)$ into the operator algebra $\mathcal{L}(S'_n)$. In particular, we have that*

$$\mu_v(1_{\mathbb{R}^m}) = (\cdot)_{S'_n}.$$

Proof. We have only to prove that

$$\mu_v(fg) = \mu_v(f) \circ \mu_v(g),$$

for every pair of functions (f, g) . Since the above operators are linear and continuous (that is Schwartz linear) it is sufficient to prove that the operators $\mu_v(fg)$ and $\mu_v(f) \circ \mu_v(g)$ are equal over one Schwartz basis, in particular the basis v itself. This is obvious, since

$$\mu_v(fg)(v) = (fg)v,$$

as we already have observed in the above property and

$$\mu_v(f) \circ \mu_v(g)(v_p) = \mu_v(f)(g(p)v_p) = g(p)f(p)v_p,$$

for every point index p , for the same reason. ■

Remark. By the way, we note that - by the above property - each spectral distribution μ_v , corresponding to a Schwartz basis, is an operator valued character of the commutative algebra $\mathcal{O}_M(\mathbb{R}^m)$.

3.2 Operator valued spectral distributions

Let us give the formal definition of our operator valued spectral distributions.

Definition (operator valued generalized measure). *We define **operator valued spectral distribution** (or **generalized spectral measure**) of \mathbb{R}^m into $\mathcal{L}(\mathcal{S}'_n)$ any operator*

$$\mu : \mathcal{O}_M^{(m)} \rightarrow \mathcal{L}(\mathcal{S}'_n)$$

which is continuous with respect to the standard topology of the function space $\mathcal{O}_M^{(m)}$ and to the pointwise topology of the operator space $\mathcal{L}(\mathcal{S}'_n)$.

Open problem. If μ is an operator valued spectral distribution of $\mathcal{O}_M^{(m)}$ into $\mathcal{L}(\mathcal{S}'_n)$, is it possible to find a Schwartz basis v (indexed by \mathbb{R}^m) such that $\mu_v = \mu$?

3.3 Integral with respect to spectral distributions

We should, now, only define a suitable integral associated with any such generalized measure, but this is straightforward; following the Radon measure convention:

- **Definition (integral with respect to an operator valued generalized measure).** *We shall put*

$$\int_{\mathbb{R}^m} f \mu := \mu(f),$$

*for every function f in $\mathcal{O}_M^{(m)}$ and we shall call the value $\mu(f)$ **integral of the function f with respect to the generalized measure μ .***

Remark. Note that - at least for the moment - the juxtaposition $f\mu$ is not a genuine product among the function f and the operator μ .

3.4 Spectral product of operators by \mathcal{S} families

But we should and can go further.

Definition (of spectral product among operators and Schwartz families). *Let v be a Schwartz family in \mathcal{S}'_n indexed by \mathbb{R}^m and let B be a linear continuous operator from \mathcal{S}'_n into \mathcal{S}'_m . We define **spectral product of the operator B by the family v** , denoted by $(B.v)$, as the operator-valued distribution*

$$(B.v) : \mathcal{O}_M^{(m)} \rightarrow \mathcal{L}(\mathcal{S}'_n) : f \mapsto \int_{\mathbb{R}^m} (fB)v,$$

where fB is the product defined by

$$fB : \mathcal{S}'_n \rightarrow \mathcal{S}'_m : (fB)(u) = fB(u),$$

and the superposition

$$\int_{\mathbb{R}^m} (fB)v$$

is the superposition of the Schwartz family v with respect to the operator fB .

With the above new definition we can write, eventually:

$$\int_{\mathbb{R}^m} (f[.|v])v = \int_{\mathbb{R}^m} f([.|v].v),$$

where the left hand side is a superposition of the Schwartz family v with respect to the operator $f[.|v]$ and the right hand side is the integral of the function f with respect to the operator valued distribution $([.|v].v)$ (spectral product of the coordinate operator of v by the family v itself).

3.5 Expansions by generalized spectral measures

Using the spectral product, our initial Schwartz spectral expansion theorem can be written in integral form as

$$A = \int_{\mathbb{R}^m} a([.|v].v).$$

Note moreover that the formal product notation $a([.|v].v)$, appearing in the above integral, can be (in a standard way) viewed - always - as a genuine authentic product. Indeed,

- define (in a standard way) the **product of a function g in $\mathcal{O}_M^{(m)}$ by a generalized measure $\mu : \mathcal{O}_M^{(m)} \rightarrow \mathcal{L}(S'_n)$ as the operator $g.\mu$ given by**

$$(g.\mu)(f) = \mu(gf),$$

for any function f in $\mathcal{O}_M^{(m)}$.

The above definition is correctly given, since the product of two slowly increasing functions in slowly increasing too.

- Define (in a standard way) the **integral of a generalized measure** as the value of the measure at the unitary constant function $1_{\mathbb{R}^m}$, that is as it follows

$$\int_{\mathbb{R}^m} \mu := \int_{\mathbb{R}^m} 1_{\mathbb{R}^m} \mu.$$

Then, the product $a.([.|v].v)$ is well defined and we have

$$\begin{aligned} \int_{\mathbb{R}^m} a.([.|v].v) &= a.([.|v].v)(1_m) = \\ &= ([.|v].v)(a) = \\ &= \int_{\mathbb{R}^m} a([.|v].v). \end{aligned}$$

So the final version of the spectral expansion theorem is

$$A = \int_{\mathbb{R}^m} a.([.|v].v),$$

and in the above integral does not appear any “formal” product but only genuine authentic algebraic products.

3.6 Generalized spectral measure on eigenspectra

Now we desire to see spectral measures from another point of view.

In the conditions of our spectral expansion theorem, consider the vector space $\mathcal{O}_M^{(a)}$ of all complex functions $f : S \rightarrow \mathbb{C}$ defined on the eigenvalue spectrum $S = a(\mathbb{R}^m)$ of the operator A and such that the composite function $f \circ a$ is a function belonging to the space $\mathcal{O}_M^{(m)}$.

We define, for every distribution u in \mathcal{S}'_n , the operator

$$\mu_a(u, v) : \mathcal{O}_M^{(a)} \rightarrow \mathcal{S}'_m : f \mapsto (f \circ a)[u|v].$$

Note that, if j_S is the canonical immersion of the eigenvalue spectrum S into the complex plane, we have immediately

$$\mu_a(u, v)(j_S) = (j_S \circ a)[u|v] = a[u|v],$$

so that the value of the operator $\mu_a(u, v)$ at the immersion j_S is the a -multiple of the coordinate distribution of u in the basis v . We so have immediately the following superposition expansion

$$A(u) = \int_{\mathbb{R}^m} \mu_a(u, v)(j_S)v,$$

for every u in \mathcal{S}'_n . But we can go further.

- **Position of the problem.** *We desire to see this operator $\mu_a(u, v)$ as a generalized measure (always in the sense of linear and continuous operator) on the eigenspectrum S .*

To reach our aim we should define a right topology on the space $\mathcal{O}_M^{(a)}$ and this can be do in a standard a natural way.

Topology on $\mathcal{O}_M^{(a)}$. Let a be a slowly increasing function belonging to the space $\mathcal{O}_M^{(m)}$. The function

$$J_a : \mathcal{O}_M^{(a)} \rightarrow \mathcal{O}_M^{(m)} : f \mapsto f \circ a,$$

is a linear injection that we call the natural injection of $\mathcal{O}_M^{(a)}$ into $\mathcal{O}_M^{(m)}$, so $\mathcal{O}_M^{(a)}$ is linearly isomorphic with the subspace $J_a(\mathcal{O}_M^{(a)})$ of the space $\mathcal{O}_M^{(m)}$. Thus $\mathcal{O}_M^{(a)}$ can inherit naturally the topology of its linearly isomorphic image $J_a(\mathcal{O}_M^{(a)})$ via the injection J_a ; a subset O of V_a is open in this topology if and only if its image $J_a(O)$ - by the injection J_a - is open in $J_a(\mathcal{O}_M^{(a)})$. This is equivalent to say that we consider open only those sets of $\mathcal{O}_M^{(a)}$ which are the reciprocal image by the injection J_a of a open sets of the space $\mathcal{O}_M^{(m)}$. Equivalently, we endow the space $\mathcal{O}_M^{(a)}$ with the coarsest topology making the injection J_a continuous (the so called initial topology relative to the injection J_a).

In the above conditions the operator

$$\mu_a(u, v) : \mathcal{O}_M^{(a)} \rightarrow \mathcal{S}'_m : f \mapsto (f \circ a)[u|v],$$

is the composition $M_{[u|v]} \circ J_a$ of the continuous linear injection $J_a : \mathcal{O}_M^{(a)} \rightarrow \mathcal{O}_M^{(m)}$ with the continuous linear multiplicative operator

$$M_{[u|v]} : \mathcal{O}_M^{(m)} \rightarrow \mathcal{S}'_m : g \mapsto g[u|v],$$

consequently the operator $\mu_a(u, v)$ is linear and continuous and so it is a genuine generalized measure in our following sense.

3.7 Measures on eigenspectra and their integrals

Definition (of generalized measure on parametrized spectra). *Let a be a smooth slowly increasing function belonging to $\mathcal{O}_M^{(m)}$ and let S be its image. We define **generalized measure on the set S** , or more precisely **on the parametrization a of the set S** , every linear continuous operator defined on the space $\mathcal{O}_M^{(a)}$ endowed with its natural topology.*

Definition (integral of a generalized measure). *For such generalized measures on S , say $\mu : \mathcal{O}_M^{(a)} \rightarrow \mathcal{S}'_m$, we define the integral of μ over S the distribution*

$$\int_S \mu = \mu(1_S),$$

*where 1_S is the constant unitary function on S . Moreover, for each function f in the algebra $\mathcal{O}_M^{(a)}$ we define **integral of f with respect to the generalized measure μ** the value $\mu(f)$ of μ at f .*

The definition is well given, indeed every constant function defined on the spectrum S lives in $\mathcal{O}_M^{(a)}$, so it is, in particular, for the unitary constant function 1_S .

Moreover, we give the following.

Definition (product of a generalized measure on a by functions in $\mathcal{O}_M^{(a)}$). *We define product of any function f in the space $\mathcal{O}_M^{(a)}$ by a generalized measure $\mu : \mathcal{O}_M^{(a)} \rightarrow \mathcal{S}'_m$ as the operator $f\mu$, from $\mathcal{O}_M^{(a)}$ into \mathcal{S}'_m , defined by*

$$(f\mu)(g) = \mu(fg),$$

for every g in $\mathcal{O}_M^{(a)}$.

Remark. Note that - as we have already said - the space $\mathcal{O}_M^{(a)}$ is an algebra with respect to the pointwise standard operations. Indeed, if f and g lie in the subspace $\mathcal{O}_M^{(a)}$, then

$$\begin{aligned} ((fg) \circ a)(p) &= (fg)(a(p)) = \\ &= f(a(p))g(a(p)) = \\ &= (f \circ a)(p)(g \circ a)(p) = \\ &= (f \circ a)(g \circ a)(p), \end{aligned}$$

and the product of the two $\mathcal{O}_M^{(m)}$ functions $f \circ a$ and $g \circ a$ lies yet in the space $\mathcal{O}_M^{(m)}$.

- The value of a generalized measure μ at a function f in $\mathcal{O}_M^{(a)}$ (that is the integral of a function f in $\mathcal{O}_M^{(a)}$ with respect to the measure μ) can be viewed as the integral of the measure $f\mu$.

Indeed, we have

$$\int_S f\mu = (f\mu)(1_S) = \mu(f),$$

for every f in $\mathcal{O}_M^{(a)}$.

3.8 Expansions by integration on eigenspectra

Concluding, we obtain another meaningful form of the spectral expansion theorem.

Theorem. *In the conditions of our spectral expansion theorem, we have that the coordinate system of the image $A(u)$ of any tempered distribution with respect to the Schwartz eigenbasis v of the operator A itself can be expanded as an integral of generalized spectral measure; specifically we have*

$$[A(u)|v] = \int_S j_S \mu_a(u, v),$$

for every tempered distribution u .

Proof. Indeed, we have

$$\begin{aligned}
[A(u)|v] &= \left[\int_{\mathbb{R}^m} (a[u|v])v|v \right] = \\
&= a[u|v] = \\
&= \mu_a(u, v)(j_S) = \\
&= \int_S j_S \mu_a(u, v),
\end{aligned}$$

for every tempered distribution u in \mathcal{S}'_n . ■

3.9 Operator valued distributions on eigenspectra

Consider now the operator

$$\mu_a[., v] : \mathcal{O}_M^{(a)} \rightarrow \mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m) : f \mapsto (f \circ a)[.|v].$$

This new operator can be considered also as a generalized measure on the parametric spectrum a , so we have

$$\int_S j_S \mu_a[., v] = (j_S \circ a)[.|v] = a[.|v],$$

and

$$\int_S \mu_a[., v] = (1_S \circ a)[.|v] = [.|v].$$

Consider moreover the operator

$$\mu_{(a,v)} : \mathcal{O}_M^{(a)} \rightarrow \mathcal{L}(\mathcal{S}'_n) : f \mapsto \int_{\mathbb{R}^m} (f \circ a)[.|v]v.$$

Also this operator can be considered as a generalized measure on the parametric spectrum a , so we have

$$\begin{aligned}
\int_a j_S \mu_{(a,v)} &= \int_{\mathbb{R}^m} (j_S \circ a)[.|v]v = \\
&= \int_{\mathbb{R}^m} a[.|v]v = \\
&= A,
\end{aligned}$$

and

$$\begin{aligned}
\int_a \mu_{(a,v)} &= \int_{\mathbb{R}^m} (1_S \circ a)[\cdot|v]v = \\
&= \int_{\mathbb{R}^m} [\cdot|v]v = \\
&= (\cdot)_{\mathcal{S}'_n}.
\end{aligned}$$

Exactly the analogous of the definition of spectral measure we gave in the beginning of the section where the space $\mathcal{O}_M^{(a)}$ is instead of the space $C^0(S)$.

Definition (of spectral distribution of an operator). *Each operator of the above form $\mu_{(a,v)}$, where the pair (a, v) is a Schwartz eigensolution of the operator A (this means simply that v is a Schwartz eigenbasis of A and a is the corresponding system of eigenvalues) is called a **spectral distribution of the operator A** .*

4 \mathcal{S} Expansions and \mathcal{S} linear equations

Let A be an \mathcal{S} linear operator on the space \mathcal{S}'_n and let v be an \mathcal{S} basis of the space \mathcal{S}'_n such that $Av = av$, with a a function of class \mathcal{O}_M . We desire to solve the \mathcal{S} linear equation

$$E : A(\cdot) = d,$$

with d in \mathcal{S}'_n .

Theorem. *Let A be an \mathcal{S} linear operator on the space \mathcal{S}'_n and let v be an \mathcal{S} basis of the space \mathcal{S}'_n , indexed by the m -dimensional Euclidean space, such that $Av = av$, with a a function of class \mathcal{O}_M . Then, the \mathcal{S} linear equation*

$$E : A(\cdot) = d,$$

with d in \mathcal{S}'_n , admits (at least) one solution if and only if the representation d_v , of the datum d in the \mathcal{S} basis v , is divisible by the function a . In this case, a solution of the equation E is the representation of any quotient q , of the division of d_v by a , in the inverse basis of v , that is the superposition

$$\int_{\mathbb{R}^m} qv.$$

Proof. (\Rightarrow) Let u be a solution of the equation E . We have

$$A(u) = \int_{\mathbb{R}^m} a[u|v]v,$$

by the spectral \mathcal{S} expansion theorem and

$$d = \int_{\mathbb{R}^m} [d|v]v,$$

by the definition of representation of d in the basis v . Since v is \mathcal{S} linear independent, we obtain the eigen-representation of the equality $E(u) : A(u) = d$, that is the equality

$$a[u|v] = [d|v],$$

so that, the distribution $d_v = [d|v]$ is divisible by the function a , since there exists a distribution q such that

$$aq = d_v.$$

(\Leftarrow) Vice versa, if the representation d_v is divisible by the function a , then any quotient q of the division of d_v by a is a solution of E . Indeed, let q such a quotient, we have

$$\begin{aligned} A\left(\int_{\mathbb{R}^m} qv\right) &= \int_{\mathbb{R}^m} qA(v) = \\ &= \int_{\mathbb{R}^m} q(av) = \\ &= \int_{\mathbb{R}^m} (aq)v = \\ &= \int_{\mathbb{R}^m} d_v v = \\ &= d, \end{aligned}$$

as we claimed. ■

We can see an interesting application.

Application (the Malgrange-Ehrenpreis theorem). We obtain, as a very particular case the Malgrange theorem, using the Hörmander division of

a distribution by polynomials. First of all consider that the partial derivative ∂_i has the Fourier basis as an \mathcal{S} eigenfamily, indeed we have

$$\partial_i(e^{-i(p|\cdot)}) = -ip_i e^{-i(p|\cdot)},$$

for every positive integer i less than n . Consequently we have

$$\partial^j(e^{-i(p|\cdot)}) = (-i)^{|j|} p^j e^{-i(p|\cdot)},$$

for every multi-index j ; thus a differential operator D with constant coefficients, say

$$D = \Sigma c_j \partial^j,$$

has the Fourier basis $v = (e^{-i(p|\cdot)})_{p \in \mathbb{R}^m}$ as an \mathcal{S} eigenbasis. If q is the quotient of the division of a distribution d by a polynomial $\Sigma(-i)^{|j|} c_j(\cdot)^p$, the \mathcal{S} linear equation

$$Du = q$$

has the solution

$$\int_{\mathbb{R}^m} qv,$$

by the above theorem, and this is exactly what the Malgrange theorem says.

5 Existence of Schwartz Green families

Theorem. *Let $L \in \mathcal{L}(\mathcal{S}'_n)$ be an \mathcal{S} linear operator. Let λ be an \mathcal{S} eigenfamily of the operator L with corresponding eigenvalue system l , i.e. let the equality*

$$L(\lambda_p) = l(p)\lambda_p,$$

hold true, for every p in the index set, say I , of the family λ . Assume that

- *there is another \mathcal{S} family μ such that the Dirac family of the space \mathcal{S}'_n can be factorized as the product*

$$\mu \cdot \lambda = \delta,$$

in other terms assume that the Schwartz family λ has an \mathcal{S} left inverse with respect to the product of Schwartz families;

- the function l is an \mathcal{O}_M function, it is nowhere zero and its inverse l^{-1} is of class \mathcal{O}_M too, that is we require that l is an invertible element of the ring \mathcal{O}_M .

Then, the operator L has an \mathcal{S} Green family, namely the family G defined by

$$G_p = \int_{\mathbb{R}^n} (1/l) \mu_p \lambda,$$

for every index p in I .

Proof. Indeed, for every index p , we have

$$\begin{aligned} L(G_p) &= L \left(\int_{\mathbb{R}^n} (1/l) \mu_p \lambda \right) = \\ &= \int_{\mathbb{R}^n} (1/l) \mu_p L(\lambda) = \\ &= \int_{\mathbb{R}^n} (1/l) \mu_p (l\lambda) = \\ &= \int_{\mathbb{R}^n} l(1/l) \mu_p \lambda = \\ &= \int_{\mathbb{R}^n} \mu_p \lambda = \\ &= \delta_p, \end{aligned}$$

as we claimed. ■

The above assumptions imply that the family λ is a system of \mathcal{S} generators for \mathcal{S}'_n and that the family μ is \mathcal{S} linearly independent. In the particular case in which the power $\mu.\mu$ is the factorization of the Dirac basis we deduce that the Schwartz family μ must be a Schwartz basis too.

Let us generalize the preceding result.

Theorem. *Let $L \in \mathcal{L}(\mathcal{S}'_n)$ be an \mathcal{S} linear operator with an \mathcal{S} eigenfamily λ and corresponding eigenvalue system l . Assume that*

- there is another \mathcal{S} family μ such that

$$\mu.\lambda = \delta,$$

- any member of the family μ is divisible by the function l , that is there is a family ν of distributions such that

$$l\nu_p = \mu_p,$$

for every index p of I (ν_p is the quotient of the division of μ_p by l).

Then,

- the operator L has a Green family, namely the family G defined by

$$G_p = \int_{\mathbb{R}^n} \nu_p \lambda,$$

for every index p .

- If, moreover, the family ν is of class \mathcal{S} , the operator L has an \mathcal{S} Green family, namely the family defined by the product of \mathcal{S} families $G = \nu.\lambda$.

Proof. Indeed, for every index p , we have

$$\begin{aligned} L(G_p) &= L\left(\int_{\mathbb{R}^n} \nu_p \lambda\right) = \\ &= \int_{\mathbb{R}^n} \nu_p L\lambda = \\ &= \int_{\mathbb{R}^n} \nu_p (l\lambda) = \\ &= \int_{\mathbb{R}^n} (l\nu_p) \lambda = \\ &= \int_{\mathbb{R}^n} \mu_p \lambda = \\ &= \delta_p, \end{aligned}$$

as we claimed. ■

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